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Midterm

The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics.
Total point 220 pt.

Problem 1 : (50 pt)
For the system $3y + 3z = 6, x + 2z = 3, x - y + z = 1$, do the following :

1. **Write the system in the matrix equation form.**
2. **Write the system in the vector equation form.**
3. **Write the augmented matrix and row reduce it.**
4. **Write out the complete set of solutions (if they exist) in vector form using parameters if needed.**
5. **Describe the solution set geometrically.**

Solution :

1. Let $A = \begin{pmatrix} 0 & 3 & 3 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$. Then, the matrix equation associated to the system is $Ax = b$.
2. Let $v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Then the vector equation associated to the system is

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$$

3. The augmented matrix associated to the system is

$$\begin{pmatrix} 0 & 3 & 3 & 6 \\ 1 & 0 & 2 & 3 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

We row reduce this matrix :

$$\begin{pmatrix} 0 & 3 & 3 & 6 \\ 1 & 0 & 2 & 3 \\ 1 & -1 & 1 & 1 \end{pmatrix} \sim^{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & 3 & 6 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

$$\sim^{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

$$\sim^{R_2 \leftarrow 1/3 R_2} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 3 & 3 & 6 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

$$\sim^{R_2 \leftarrow 1/3 R_2} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

$$\sim^{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

$$\sim^{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4. The system is consistent since the last column of the augmented matrix is not a pivot column. For the reduced echelon form obtained in the previous question, we obtain the system :

$$\begin{cases} x_1 + 2x_3 = 3 \\ x_2 + x_3 = 2 \\ 0 = 0 \end{cases}$$

Since there is a free variable x_3, then the system has infinitely many solution. We express the basic variables in terms of the free variable and we get that the general form of a solution is :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

where $t \in \mathbb{R}$.

5. Geometrically, the solution set is a line passing through $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$ with direction $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$.

Problem 2 : (20 pt)
Balance the chemical equation :

$$NH_3 + O_2 \to N_2 + H_2O$$

Solution : We want to balance the chemical equation. We write the molecule into a vector with the Nitrate in first position, Hydrogen in second position and Oxygen in third position.

$$NH_3 : \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, O_2 : \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, N_2 : \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \textit{ and } H_2O : \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

Balancing the chemical equation is equivalent to find $r, s, t, u \in \mathbb{R}$ such that :

$$r \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - u \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0$$

The augmented matrix associated to the system is :

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{pmatrix}$$

We can row reduce the augmented matrix.

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{pmatrix} \sim_{R_2 \leftarrow R_2 - 3R_1} \begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{pmatrix}$$

$$\sim_{R_3 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 6 & -2 & 0 \end{pmatrix}$$

$$\sim_{R_2 \leftarrow 1/2 R_2 \textit{ and } R_3 \leftarrow 1/6 R_3} \begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{pmatrix}$$

$$\sim_{R_1 \leftarrow R_1 + 2R_3} \begin{pmatrix} 1 & 0 & 0 & -2/3 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{pmatrix}$$

From the reduced echelon form, we obtain the system :

$$\begin{cases} r - 2/3u = 0 \\ s - 1/2u = 0 \\ t - 1/3u = 0 \end{cases}$$

So that, the general form of a solution of this system is :

$$\begin{pmatrix} r \\ s \\ t \\ u \end{pmatrix} = x \begin{pmatrix} 2/3 \\ 1/2 \\ 1/3 \\ 1 \end{pmatrix}$$

with $x \in \mathbb{R}$.
For instance, taking $u = 6$, we get

$$\begin{pmatrix} r \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 6 \end{pmatrix}$$

as a particular solution. So that a balanced chemical equation is

$$4NH_3 + 3O_2 \to 2N_2 + 6H_2O$$

Problem 3 : (60 pt)
Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $x \mapsto Ax$.

1. **Use the algorithm seen in class, to determine if A is invertible and if it is to compute its inverse. Deduce the inverse of the linear transformation T ?**
2. **Explain what does it means for the columns of A to span $\mathbb{R}^2$. Does the columns of A span $\mathbb{R}^3$?**
3. **Explain what does it means for the columns of A to be linearly independent. Are the columns of A linearly independent ?**
4. **Recall the definition of an onto $\mathbb{R}^2$ transformation and also the one of a one-to-one transformation. Deduce from the previous question if T is one-to-one and then if T is onto $\mathbb{R}^3$.**
5. **Find the image of $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.**
6. **Find a $x \in \mathbb{R}^2$, such that its image by T is $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.**

Solution :

1. We know that for the class that $[A, I_2]$, row reduce to $[I_2, A^{-1}]$.

$$\begin{aligned} [A, I_2] &= \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \\ &\sim_{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \\ &\sim_{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix} \end{aligned}$$

So that

$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

And T^{-1} is the linear transformation whose standard matrix is A^{-1}. So that T^{-1} maps x into $A^{-1}x$.

2. The column of A, c_1 and c_2 span $\mathbb{R}^2$ if for each $b \in \mathbb{R}^2$, the equation $Ax = b$ has at least a solution. Since we have seen that the reduced echelon form of A is I_2, from the last question. We have a pivot position for A in each row, so the column of A span $\mathbb{R}^2$.
3. The columns of A are linearly independent if the equation $Ax = 0$ has only the trivial solution. Since the columns of A are not multiple of each other, then the columns of A are linearly independent (this criterion works because we are dealing only with two columns).
4. A transformation is onto $\mathbb{R}^2$ if there is at least a x such that $T(x) = Ax = b$, for each $b \in \mathbb{R}^2$. So from **2.**, we know that T is onto. The transformation is one-to-one if there is at most a solution x for the system $T(x) = Ax = b$, for each $b \in \mathbb{R}^2$. That is equivalent to requiring that the equation $Ax = 0$ has only the trivial solution as a unique solution. From **3.**, T is one-to-one.
5. The image of $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is
$$T(b) = Ab = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$
6. We are looking for $x \in \mathbb{R}^2$, such that $T(x) = Ax = c$. From a theorem of the class, we know that the equation $Ax = c$ has a unique solution :
$$x = A^{-1}c = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Problem 4 : (40 pt)

1. **Write the standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points through the x_1-axis.**
2. **Write the standard matrix B of the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points through the x_2-axis.**
3. **Express the standard matrix of the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points through the x_1-axis and that reflects points through the x_2-axis in term of A and B and compute it.**
4. **Deduce from the previous question that R is a rotation through the origin. What is the angle of this rotation ?**

Solution :

1. The standard matrix A of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points through the x_1-axis is
$$A = [T(e_1), T(e_2)]$$
$$T(e_1) = e_1,\ T(e_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

So that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. The standard matrix B of the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points through the x_2-axis is

$$B = [S(e_1), S(e_2)]$$

$$T(e_1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So that

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

3. The standard matrix of the linear transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects points through the x_1-axis and that reflects points through the x_2-axis which is indeed the composite $S \circ T$, is

$$BA = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

4. We notice that S is a rotation through the origin of angle π. Indeed, the standard matrix for a rotation of angle θ is of the form

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

and we notice that

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $\theta = \pi$ so the two transformation have same standard matrix and thus they are the same.

Problem 5 : (30 pt)

Let $A = \begin{pmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{pmatrix}$ **and** $b = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$**. The LU factorization of** A **is**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Use the LU factorization to solve the system $Ax = b$**.**

Solution : Let $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{pmatrix}$. We have $Ax = LUx = b$. In order to solve this equation, set $Ux = y$, we first solve the equation $Ly = b$ and find y and then $Ux = y$ and find x and since $Ax = LUx = Ly = b$ then x will be also solution of $Ax = b$. The augmented matrix associated to the system $Ly = b$

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 6 \\ -2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 6 \end{array}\right) \sim^{R_2 \leftarrow R_2 + 2R_1 \text{ and } R_3 \leftarrow R_3 - 3R_1} \left(\begin{array}{cccc} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 12 \\ 0 & -1 & 1 & -12 \end{array}\right)$$

$$\sim^{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{cccc} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

As a conclusion, from the reduced echelon form, we get that the only solution of this system is $y = \begin{pmatrix} 6 \\ 12 \\ 0 \end{pmatrix}$. Now, we solve the system $Ux = y$. The augmented matrix associated to this system is

$$\left(\begin{array}{cccc} 2 & -4 & 2 & 6 \\ 0 & -3 & 6 & 12 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim^{R_1 \leftarrow 1/2R_1 \text{ and } R_2 \leftarrow -1/3R_2} \left(\begin{array}{cccc} 1 & -2 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim^{R_1 \leftarrow R_1 + 2R_2 \text{ and } R_2 \leftarrow R_2 + 3R_3} \left(\begin{array}{cccc} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim^{R_1 \leftarrow R_1 + 2R_2} \left(\begin{array}{cccc} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

So, form this last echelon form, we get thaat the only solution of the system $Ux = y$ and thus the one of the system $Ax = b$ is

$$x = \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix}$$

Problem 6 : (20 pt)
Let $A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -3 & -5 \\ 2 & 1 \end{pmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.
Solution :

$$AB = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -21 & -21 \\ 7 & 7 \end{pmatrix}$$

and

$$AC = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -21 & -21 \\ 7 & 7 \end{pmatrix}$$

So $AB = AC$, (both are matrices of same size with same entries) but $B \neq C$ since they have same size but different entries.